



# Geometric Modeling of Inclusions as Ellipsoids

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# Geometric Modeling of Inclusions as Ellipsoids

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## Abstract

Nonmetallic inclusions in gas turbine disk alloys can have a significant detrimental impact on fatigue life. Because large inclusions that lead to anomalously low lives occur infrequently, probabilistic approaches can be utilized to avoid the excessively conservative assumption of lifing to a large inclusion in a high stress location. A prerequisite to modeling the impact of inclusions on the fatigue life distribution is a characterization of the inclusion occurrence rate and size distribution. To help facilitate this process, a geometric simulation of the inclusions was devised. To make the simulation problem tractable, the irregularly sized and shaped inclusions were modeled as arbitrarily oriented, three independent dimensioned, ellipsoids. Random orientation of the ellipsoid is accomplished through a series of three orthogonal rotations of axes. In this report, a set of mathematical models for the following parameters are described: the intercepted area of a randomly sectioned ellipsoid, the dimensions and orientation of the intercepted ellipse, the area of a randomly oriented sectioned ellipse, the depth and width of a randomly oriented sectioned ellipse, and the projected area of a randomly oriented ellipsoid. These parameters are necessary to determine an inclusion's potential to develop a propagating fatigue crack. Without these mathematical models, computationally expensive search algorithms would be required to compute these parameters.

## Introduction

Nonmetallic inclusions (NMI) in powder metallurgy (PM) alloys tend to cause early formation of propagating cracks that can lead to catastrophic failures in gas turbine engine rotating components. These NMIs tend to be irregularly shaped although mostly convex (Fig. 1). They also can be deformed, broken, and oriented (although not perfectly so) by the processing deformations (extrusion and forging). In order to model the crack initiating potential of these inclusions, ellipsoids may be used as an approximation of the inclusion's size and shape. A series of mathematical models for section and projection parameters of randomly oriented general (three independent semi-axis dimensions) ellipsoids have been derived. Although the parameters described by these solutions can be determined using iterative search algorithms, these tend to be computationally expensive. The "closed-form" mathematical models speed the execution of Monte Carlo simulations of inclusion populations in test specimens and prototype components (Ref. 1).

A parameter that can be used to assess an inclusion's potential of initiating a propagating fatigue crack is the area perpendicular to the principal loading direction. A solution for the intercepted area (planar section) of oblate and prolate spheroids (two independent dimensions; rotational symmetry about one semi-axis, often referred to as "regular" ellipsoids) was derived by Dehoff in 1962 (Ref. 2). No such solution could be found for general (three independent semi-axes or 3-D) ellipsoids. A solution for the intercepted area of an arbitrarily oriented general ellipsoid was derived with the assistance of symbolic math software and will be presented herein. Additionally, solutions for the dimensions and orientation of the intercepted ellipse were derived as they could also potentially provide correlations with observations of inclusion initiated cracks. These solutions were necessary for comparison with the distributions of the dimensions and orientations of inclusions observed on polished sections of turbine disk forgings (Ref. 1).



Figure 1.—Crack emanating from a ceramic inclusion as observed on a smooth bar fatigue specimen.

An crack initiating inclusion connected to the component surface can be envisioned as being sectioned twice; once by the surface and again by the crack. With the ellipsoid approximation, the resulting area on the fracture surface is a sectioned ellipse. A solution for the area, as well as the depth and width, of an arbitrarily sized and oriented sectioned ellipse were derived.

Observations of fracture surfaces and cracked inclusions on specimen surfaces has led to the proposition that the projected area (perpendicular to the maximum principal stress) of an inclusion may be a better indicator of the inclusion's crack initiating potential. A solution for the projected area of an arbitrarily oriented 3-D ellipsoid is also provided.

The solutions described in the following sections have proved useful in simulating the area distributions of inclusions randomly distributed in a specimen volume. Other applications may benefit from the reduction in computational effort required in utilizing search algorithms often used in computing the cross sectional and projected area of an ellipsoid.

### **Intercepted Area of a General Ellipsoid (Three Independent Semi-Axes) Cut by an Arbitrary Plane**

A search of the literature for a closed form solution for the intersection between a plane and a general (three independent semi-axes) ellipsoid was not productive. The following describes a solution for the intercepted area of general ellipsoid.

For a triaxial or general ellipsoid, a sequence of three rotations is necessary to arrive at an arbitrary orientation (Fig. 2) (Ref. 3). Only two rotations are necessary for a regular ellipsoid. To track the new orientation, a new local coordinate system is defined after each rotation. Choosing the first rotation to be an angle  $\phi$  about the original  $z$  axis gives new  $x$  and  $y$  coordinate axes that will be referred to as  $x'$  and  $y'$ . A second rotation of an angle  $\theta$  about the  $y'$  axis gives new coordinate axes of  $x''$  and  $z''$ . The third rotation of an angle  $\psi$  about the  $x''$  axis gives the final ellipsoid orientation. This rotation order will be referred to as a  $zyx$  rotation. This is just one of the twelve possible orthogonal rotation sequences (Ref. 3).

Given the equation of a general ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad (1)$$

with  $a$ ,  $b$ , and  $c$  being the ellipsoid semi-axes dimensions (half the ellipsoid's length, breadth and depth) which are initially aligned with the  $x$ ,  $y$ , and  $z$  axes, respectively.

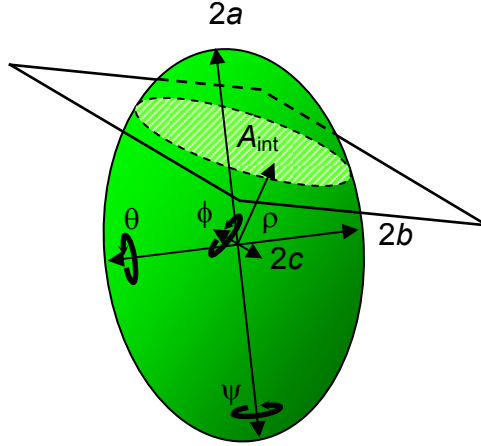


Figure 2.—Illustration of a randomly oriented general ellipsoid with intercepted area shown.

The first rotation about the  $z$  axis accomplished with the following transformation

$$\begin{aligned} x' &= x \cos \phi + y \sin \phi \\ y' &= -x \sin \phi + y \cos \phi \\ z' &= z \end{aligned} \quad (2)$$

In a similar manner, two additional rotations about the  $y'$  and  $x''$  axes can be executed. This series of rotations may be computed as a series of matrix multiplications

$$\begin{aligned} &\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \psi & \sin \psi \\ 0 & -\sin \psi & \cos \psi \end{bmatrix} \cdot \begin{bmatrix} \cos \theta & 0 & -\sin \theta \\ 0 & 1 & 0 \\ \sin \theta & 0 & \cos \theta \end{bmatrix} \cdot \begin{bmatrix} \cos \phi & \sin \phi & 0 \\ -\sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix} = \\ &\begin{bmatrix} x \cos \phi \cos \theta & y \sin \phi \cos \theta & -z \sin \theta \\ x(\cos \phi \sin \theta \sin \psi - \sin \phi \cos \psi) & y(\sin \phi \sin \theta \sin \psi + \cos \phi \cos \psi) & z \cos \theta \sin \psi \\ x(\cos \phi \sin \theta \cos \psi + \sin \phi \sin \psi) & y(\sin \phi \sin \theta \cos \psi - \cos \phi \sin \psi) & z \cos \theta \cos \psi \end{bmatrix} \end{aligned} \quad (3)$$

Substituting the thrice rotated terms into the original ellipsoid Equation (1) gives

$$\begin{aligned} &\frac{(x \cos \phi \cdot \cos \theta + y \sin \phi \cdot \cos \theta - z \sin \theta)^2}{a^2} \dots \\ &+ \frac{[x(\cos \phi \cdot \sin \theta \cdot \sin \psi - \sin \phi \cdot \cos \psi) + y(\sin \phi \cdot \sin \theta \cdot \sin \psi + \cos \phi \cdot \cos \psi) + z \cos \theta \cdot \sin \psi]^2}{b^2} \dots \\ &+ \frac{[x(\cos \phi \cdot \sin \theta \cdot \cos \psi + \sin \phi \cdot \sin \psi) + y(\sin \phi \cdot \sin \theta \cdot \cos \psi - \cos \phi \cdot \sin \psi) + z \cos \theta \cdot \cos \psi]^2}{c^2} = 1 \end{aligned} \quad (4)$$

Defining a sectioning plane as a constant distance from the origin along the original  $x$ -axis,  $x = \rho$ , the rotated Equation (4) can be rearranged into the following form

$$Ay^2 + Byz + Cz^2 + Dy + Ez + F = 0 \quad (5)$$

With some algebraic manipulation, the coefficients  $A$  to  $F$  are

$$A = \frac{\sin^2 \phi \cdot \cos^2 \theta}{a^2} + \frac{(\sin \phi \cdot \sin \theta \cdot \sin \psi + \cos \phi \cdot \cos \psi)^2}{b^2} \dots + \frac{(\sin \phi \cdot \sin \theta \cdot \cos \psi - \cos \phi \cdot \sin \psi)^2}{c^2} \quad (6)$$

$$B = 2 \cdot \cos \theta \cdot \frac{-\sin \phi \cdot \sin \theta}{a^2} + 2 \cdot \cos \theta \cdot \sin \psi \frac{\sin \phi \cdot \sin \theta \cdot \sin \psi + \cos \phi \cdot \cos \psi}{b^2} \dots + 2 \cdot \cos \theta \cdot \cos \psi \frac{\sin \phi \cdot \sin \theta \cdot \cos \psi - \cos \phi \cdot \sin \psi}{c^2} \quad (7)$$

$$C = \frac{\sin^2 \theta}{a^2} + \frac{\cos^2 \theta \cdot \sin^2 \psi}{b^2} + \frac{\cos^2 \theta \cdot \cos^2 \psi}{c^2} \quad (8)$$

$$D = 2\rho \cdot \frac{\sin \phi \cdot \cos \phi \cdot \cos^2 \theta}{a^2} \dots + 2\rho \cdot \frac{(\sin \phi \cdot \sin \theta \cdot \sin \psi + \cos \phi \cdot \cos \psi)(\cos \phi \cdot \sin \theta \cdot \sin \psi - \sin \phi \cdot \cos \psi)}{b^2} \dots + 2\rho \cdot \frac{(\sin \phi \cdot \sin \theta \cdot \cos \psi - \cos \phi \cdot \sin \psi)(\cos \phi \cdot \sin \theta \cdot \cos \psi + \sin \phi \cdot \sin \psi)}{c^2} \quad (9)$$

$$E = 2\rho \cdot \frac{-\cos \phi \cdot \cos \theta \cdot \sin \theta}{a^2} + 2\rho \cdot \cos \theta \cdot \sin \psi \frac{(\cos \phi \cdot \sin \theta \cdot \sin \psi - \sin \phi \cdot \cos \psi)}{b^2} \dots + 2\rho \cdot \cos \theta \cdot \cos \psi \frac{(\cos \phi \cdot \sin \theta \cdot \cos \psi + \sin \phi \cdot \sin \psi)}{c^2} \quad (10)$$

$$F = \rho^2 \cdot \frac{\cos^2 \phi \cdot \cos^2 \theta}{a^2} + \rho^2 \cdot \frac{(\cos \phi \cdot \sin \theta \cdot \sin \psi - \sin \phi \cdot \cos \psi)^2}{b^2} \dots + \rho^2 \cdot \frac{(\cos \phi \cdot \sin \theta \cdot \cos \psi + \sin \phi \cdot \sin \psi)^2}{c^2} - 1 \quad (11)$$

This is a general equation for the intercepted ellipse. Solving Equation (5) for  $z$  gives



$$z_1, z_2 = \begin{bmatrix} \frac{-1}{2C} \left[ By + E - \sqrt{(By + E)^2 - 4C(Ay^2 + Dy + F)} \right] \\ \frac{-1}{2C} \left[ By + E + \sqrt{(By + E)^2 - 4C(Ay^2 + Dy + F)} \right] \end{bmatrix} \quad (12)$$

These two solutions constitute the upper and lower portions of the ellipse in the  $yz$  plane. The integral of the difference between these two solutions is the area,  $\Delta$ , of the ellipse.

$$\Delta = \frac{1}{C} \int_{y_1}^{y_2} \sqrt{(By + E)^2 - 4C(Ay^2 + Dy + F)} \quad (13)$$

The limits of integration,  $y_1$  and  $y_2$ , may be determined by recognizing that the integrand must remain real over the interval. This requires that the sum of the terms under the square root remain nonnegative. Therefore, by setting the sum of the terms under the root equal to zero and solving for  $y$ , the integration limits in the  $y$  direction are

$$y_1, y_2 = \begin{bmatrix} \frac{BE - 2CD - 2\sqrt{C[AE^2 + CD^2 - BDE - F(4AC - B^2)]}}{4AC - B^2} \\ \frac{BE - 2CD + 2\sqrt{C[AE^2 + CD^2 - BDE - F(4AC - B^2)]}}{4AC - B^2} \end{bmatrix} \quad (14)$$

With the assistance of symbolic math software to solve the definite integral, one can substitute the constants ( $A, B, C, D, E$ , and  $F$ ), and apply the identity,  $\ln(-\xi) - \ln(\xi) = i\pi$ , which results in

$$\Delta = \pi abc \frac{\Phi - \rho^2}{[\Phi]^{3/2}} = \pi abc \frac{1 - \frac{\rho^2}{\Phi}}{\sqrt{\Phi}} \quad (15)$$

where

$$\begin{aligned} \Phi = & a^2 (\cos \phi \cos \theta)^2 \dots \\ & + b^2 (\cos \phi \sin \theta \sin \psi - \sin \phi \cos \psi)^2 \dots \\ & + c^2 (\cos \phi \sin \theta \cos \psi + \sin \phi \sin \psi)^2 \end{aligned} \quad (16)$$

This  $\Delta$  is the area of the intercepted ellipse in terms of the ellipsoid dimensions, the three rotation angles, and the distance from the ellipsoid centroid to the sectioning plane. It must be noted that this solution only applies for the  $zyx$  rotation order and a sectioning plane defined by a constant  $x$ . Similar solutions may be derived for different rotation orders and sectioning planes. The solutions for different rotation orders differ only in the form and sequence of trigonometric factors in the  $\Phi$  term. For this solution, one can readily note that trigonometric factors in the  $\Phi$  term are the squares of the  $x$  coefficients in the rotated frame Equation (4). This is obviously not a coincidence. This Equation (16) also has the interesting property of calculating a positive value only if the sectioning plane actually intercepts the ellipsoid.

## Orientation and Dimensions of the Intercepted Ellipse Defined by the Intersection of a Plane and an Arbitrarily Oriented General Ellipsoid

The dimensions and orientation of the ellipse defined by the intersection of a plane and a general ellipse are of interest for fracture mechanics calculations in that it is often these dimensions that are the required parameters. Starting with Equation (5) the goal is to rearrange into the form of a general ellipse equation

$$\frac{y^2}{\hat{a}^2} + \frac{z^2}{\hat{b}^2} = 1 \quad (17)$$

Where  $\hat{a}$  and  $\hat{b}$  are the dimensions of the intercepted ellipse. To get these dimensions one must first perform a rotation of axes to eliminate the  $yz$  cross term

$$\begin{aligned} A' &= A \cos^2 \alpha + B \cos \alpha \sin \alpha + C \sin^2 \alpha \\ B' &= B(\cos^2 \alpha - \sin^2 \alpha) + 2(C - A) \sin \alpha \cos \alpha \\ C' &= A \sin^2 \alpha - B \sin \alpha \cos \alpha + C \cos^2 \alpha \\ D' &= D \cos \alpha + E \sin \alpha \\ E' &= -D \sin \alpha + E \cos \alpha \\ F' &= F \end{aligned} \quad (18)$$

Setting the  $B'$  term equal to zero, applying the appropriate trigonometric identity, and rearranging gives

$$\tan(2\alpha) = \frac{B}{A - C} \quad (19)$$

This rotation of axes also provides the orientation angle of the intercepted ellipse,  $\alpha$ .

Substitution of the rotated terms gives

$$A'y^2 + C'z^2 + D'y + E'z + F = 0 \quad (20)$$

To get Equation (20) into the standard ellipse form Equation (17) one must factor the equation by completing the squares. This gives the following solutions for the ellipse dimensions

$$\hat{a}, \hat{b} = \begin{bmatrix} \sqrt{\frac{\left(\frac{D'^2}{4A'} + \frac{E'^2}{4C'} - F\right)}{A'}} \\ \sqrt{\frac{\left(\frac{D'^2}{4A'} + \frac{E'^2}{4C'} - F\right)}{C'}} \end{bmatrix} \quad (21)$$

This solution also provides a check on the area solution derived in the previous section in that the area of the intercepted ellipse,  $\Delta$ , may also be computed by:

$$\Delta = \pi \cdot \hat{a} \cdot \hat{b} \quad (22)$$

The areas for both procedures are identical for all rotations.

### Sectioned Area of a Randomly Oriented Ellipse

The fracture area of a surface connected inclusion is a cross section of the inclusion sectioned again by the specimen surface. In the ellipsoid approximation of the inclusion this is a sectioned arbitrarily oriented ellipse (see Fig. 3).

The area of this canted ellipse cut by a line can be derived by integrating the equation of the ellipse from the leftmost edge of the ellipse to the intersecting line (constant distance from the ellipse centroid,  $r$ , Fig. 3). Starting with the equation for a rotated ellipse, where  $a$  and  $b$  are the ellipse semi-axes and  $\theta$  is the rotation angle

$$\frac{(x \cos \theta + y \sin \theta)^2}{\hat{a}^2} + \frac{(-x \sin \theta + y \cos \theta)^2}{\hat{b}^2} = 1 \quad (23)$$

and solving for  $y$

$$y_1, y_2 = \left[ \frac{x(\hat{a}^2 - \hat{b}^2) \cos \theta \sin \theta - \hat{a} \hat{b} \sqrt{\hat{a}^2 \cos^2 \theta + \hat{b}^2 \sin^2 \theta - x^2}}{\hat{a}^2 \cos^2 \theta + \hat{b}^2 \sin^2 \theta}, \frac{x(\hat{a}^2 - \hat{b}^2) \cos \theta \sin \theta + \hat{a} \hat{b} \sqrt{\hat{a}^2 \cos^2 \theta + \hat{b}^2 \sin^2 \theta - x^2}}{\hat{a}^2 \cos^2 \theta + \hat{b}^2 \sin^2 \theta} \right] \quad (24)$$

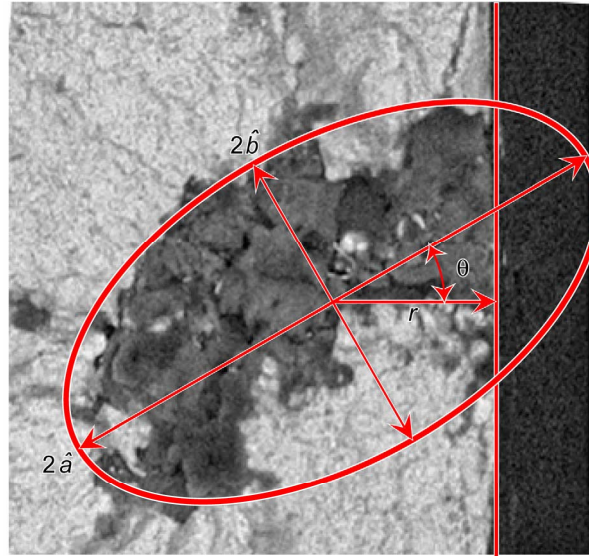


Figure 3.—Schematic of sectioned ellipse approximation of inclusion fracture area;  $\hat{a}$  and  $\hat{b}$  are ellipse semi-axes dimensions,  $r$  is the distance from the centroid to the sectioning surface, and  $\theta$  is the angle to the major axis orientation relative to the surface normal.

As shown earlier, the integrated difference between these solutions is the area of the ellipse.

$$\Delta = 2\hat{a}\hat{b} \int_{x_2}^{x_1} \frac{\sqrt{\hat{a}^2 \cos^2 \theta + \hat{b}^2 \sin^2 \theta - x^2}}{\hat{a}^2 \cos^2 \theta + \hat{b}^2 \sin^2 \theta} dx \quad (25)$$

As before, the edges of the ellipse in the  $x$  direction can be found by recognizing that the sum of the terms under the root must be nonnegative to be on the ellipse. Taking the part under the root, setting it equal to zero and solving for  $x$

$$x_1, x_2 = \left[ \begin{array}{c} \sqrt{\hat{a}^2 \cos^2 \theta + \hat{b}^2 \sin^2 \theta} \\ -\sqrt{\hat{a}^2 \cos^2 \theta + \hat{b}^2 \sin^2 \theta} \end{array} \right] \quad (26)$$

Taking the lower solution as one limit of integration and the other limit a distance,  $r$ , from the ellipse centroid (Fig. 3) gives the following definite integral

$$\Delta = 2\hat{a}\hat{b} \int_{-\sqrt{\hat{a}^2 \cos^2 \theta + \hat{b}^2 \sin^2 \theta}}^r \frac{\sqrt{\hat{a}^2 \cos^2 \theta + \hat{b}^2 \sin^2 \theta - x^2}}{\hat{a}^2 \cos^2 \theta + \hat{b}^2 \sin^2 \theta} dx \quad (27)$$

Again with the assistance of symbolic math software, this definite integral may be solved resulting in

$$\Delta = \hat{a}\hat{b} \left[ r \frac{\sqrt{\hat{a}^2 \cos^2 \theta + \hat{b}^2 \sin^2 \theta - r^2}}{\hat{a}^2 \cos^2 \theta + \hat{b}^2 \sin^2 \theta} + \frac{\pi}{2} + \arctan \left( \frac{r}{\sqrt{\hat{a}^2 \cos^2 \theta + \hat{b}^2 \sin^2 \theta - r^2}} \right) \right] \quad (28)$$

This solution for the area of a sectioned ellipse is valid for  $r$  over:  $-\sqrt{\hat{a}^2 \cos^2 \theta + \hat{b}^2 \sin^2 \theta} < r < \sqrt{\hat{a}^2 \cos^2 \theta + \hat{b}^2 \sin^2 \theta}$  which defines the interval where the solution is real, i.e., where the line intercepts the ellipse.

### **Depth and Width of a Randomly Oriented Twice-Sectioned Ellipse (Including Bounding Box Solution)**

For fracture mechanics calculations of surface connected inclusions it is necessary to know the depth and width of the feature used to approximate the crack. In this case it is the depth and width of an inclusion embedded in a metallic component as approximated by a general ellipsoid.

Given a general equation of an ellipse Equation (29) in the plane perpendicular to the loading direction (Fig. 4)

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \quad (29)$$

First the bounding box dimensions must be derived. Solving Equation (29) for  $x$

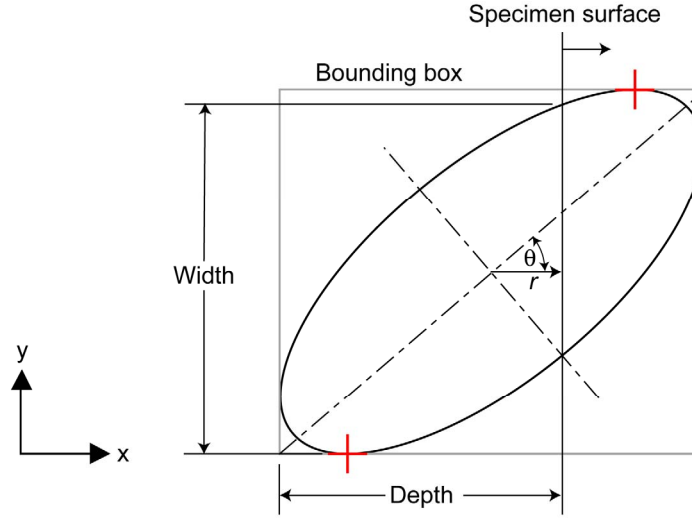


Figure 4.—Schematic description of an inclusion sectioned by the specimen surface looking perpendicular to the loading direction.

$$x_1, x_2 = \left[ \begin{array}{c} \frac{-D - By + \sqrt{y^2(B^2 - 4AC) + 2y(BD - 2AE) + D^2}}{2A} \\ \frac{-D - By - \sqrt{y^2(B^2 - 4AC) + 2y(BD - 2AE) + D^2}}{2A} \end{array} \right] \quad (30)$$

resulting in two solutions. As before, observing that the solutions are real only when the sum of the terms under the square root are nonnegative, the bounding box limits in the  $y$  direction can be solved by setting the portion under the root equal to 0 and solving for  $y$

$$y_1, y_2 = \left[ \begin{array}{c} \frac{BD - 2AE + 2\sqrt{A[AE^2 + CD^2 - BDE - F(4AC - B^2)]}}{4AC - B^2} \\ \frac{BD - 2AE - 2\sqrt{A[AE^2 + CD^2 - BDE - F(4AC - B^2)]}}{4AC - B^2} \end{array} \right] \quad (31)$$

This defines the upper and lower limits on  $y$  of the ellipse. The difference between these solutions is the bounding box dimension in  $y$

$$y_{bb} = \left[ \frac{4\sqrt{A[AE^2 + CD^2 - BDE - F(4AC - B^2)]}}{4AC - B^2} \right] \quad (32)$$

and the average of these limits is the  $y$  coordinate of the ellipse centroid

$$y_c = \frac{BD - 2AE}{4AC - B^2} \quad (33)$$

If this same procedure is performed solving first for  $y$ , similar results are found for the upper and lower limits on  $x$

$$x_1, x_2 = \left[ \frac{BE - 2CD + 2\sqrt{C\left[AE^2 + CD^2 - BDE - F(4AC - B^2)\right]}}{4AC - B^2}, \frac{BE - 2CD - 2\sqrt{C\left[AE^2 + CD^2 - BDE - F(4AC - B^2)\right]}}{4AC - B^2} \right] \quad (34)$$

Again, the difference between these solutions is the bounding box dimension in  $x$

$$x_{bb} = \left[ \frac{4\sqrt{C[AE^2 + CD^2 - BDE - F(4AC - B^2)]}}{4AC - B^2} \right] \quad (35)$$

and the average of these limits is the  $x$  coordinate of the ellipse centroid

$$x_c = \frac{BE - 2CD}{4AC - B^2} \quad (36)$$

If it is assumed that the dimensions of the inclusion that are correlated with the initial crack size are the maximum depth of the sectioned ellipsoid and the maximum width parallel to the sectioning surface (Fig. 4) then there is still some work to be done. First, to find the maximum depth, one has to add the distance from the ellipse centroid to the sectioning surface and half the bounding box dimension in  $x$  the direction (Fig. 4). The convention chosen for the distance from the intercepted ellipse centroid to the surface,  $r$ , is that  $r$  is positive when the ellipse centroid is inside the surface and negative when the ellipse centroid is outside the surface. To find the maximum width of the sectioned ellipse it must be determined if the points where the ellipse touches the bounding box are inside or outside the specimen surface. There are three possible cases (illustrated in Fig. 5). In the illustration, if the right most intercepted ellipse  $y$  bounding box tangent point is inside the surface, then the width is simply the  $y$  bounding box dimension (Fig. 5(a)). The second case would be where the surface is between the  $y$  bounding box touching points (Fig. 5(b)). In this instance the width is defined as the distance, in the  $y$  direction, from there the intercepted ellipse touches the surface to the  $y$  bounding box tangent point still inside the surface. The third instance is where the specimen surface is inside both bounding box  $y$  tangent points (Fig. 5(c)). In this case the width is the distance in  $y$  between the two points where the ellipse intersects the surface. In all three cases the depth may be defined by the addition of half the bounding box dimension in the  $x$  direction and the distance of the intercepted ellipse centroid to the specimen surface.

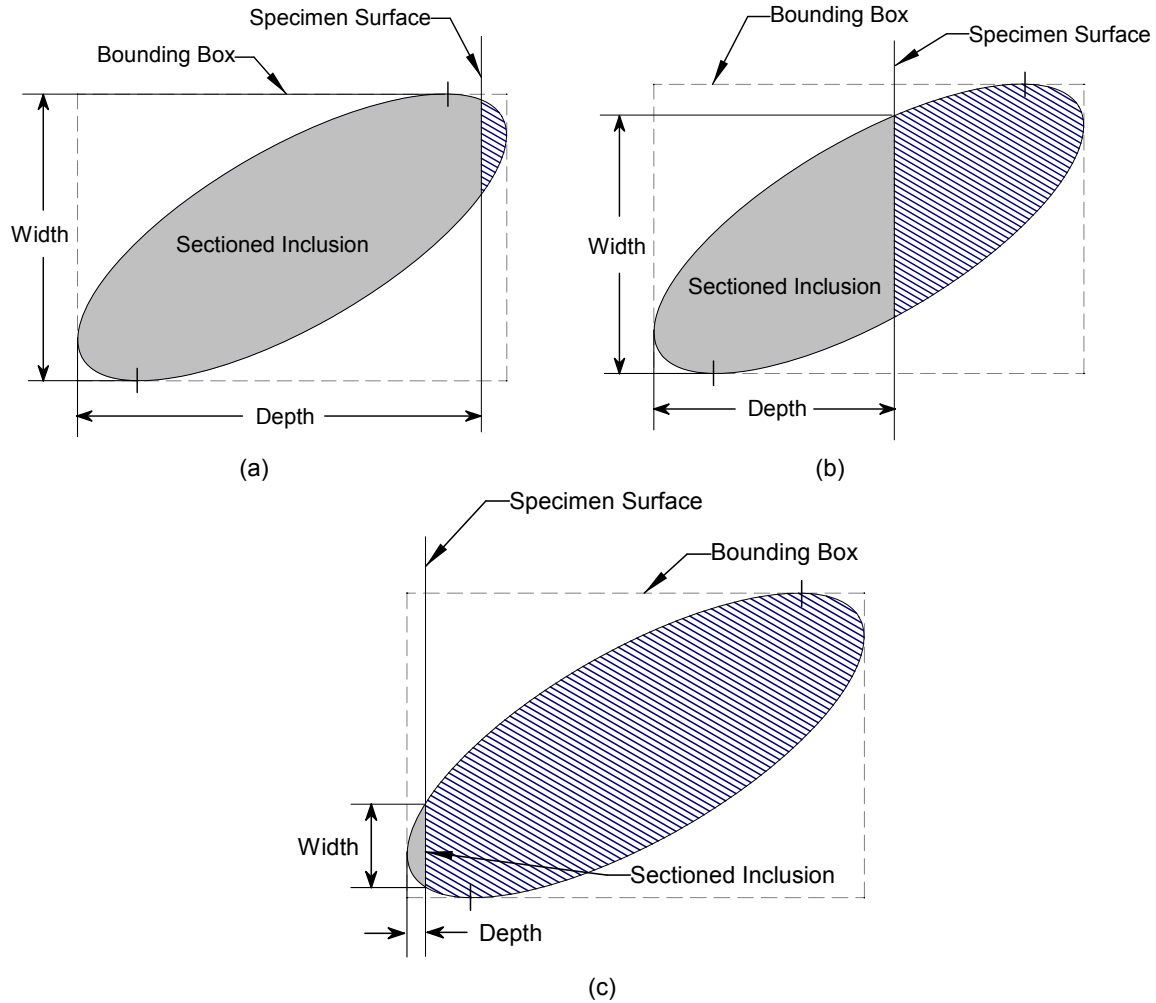


Figure 5.—Illustration of three cases of width and depth calculation for intercepted ellipsoid.

Given this information it is possible to compute the intersection points where a line (simulating the surface) intercepts the ellipse (setting  $x$  in Eq. (29) equal to  $r$  and solving for  $y$ ). It is therefore possible to compute the maximum extents of the ellipse both parallel (width) and perpendicular (depth) to the line (specimen surface). These parameters are needed to estimate the distribution of expected widths and depths of surface connected inclusions that could initiate fatigue cracks.

### Projected Area of a Randomly Oriented General Ellipsoid

The projected area may be defined as the area of the shadow of an object left on a plane that is perpendicular to an infinitely distant light source. The projected area of a randomly oriented ellipsoid may be determined by deriving an equation for the maximum distance from a line drawn through the ellipsoid centroid and perpendicular to the projection plane. The equation of this distance is the equation of the projected ellipse. Starting again with an equation of a general ellipsoid

$$Ax^2 + Bxy + Cy^2 + Exz + Dyx + Fz^2 = 1 \quad (37)$$

The  $A$  to  $F$  coefficients are functions of the ellipsoid dimensions and the rotation angles of the three rotation transformations as described earlier. Note that in this derivation the distance from the  $xy$  plane,  $z$ , is not to be included in the  $E$ ,  $D$ , and  $F$  coefficients (the  $\rho$  in Eqs. (9 to 11) would be removed) and 1 was not subtracted from both sides and included in the  $F$  term (as in Eqs. (5), (11), and (29)). The  $xy$  plane ( $z = 0$ ), in this derivation, is the projection plane.

First, a rotation of axes is performed to eliminate the  $B$  term; this gives an equation that may be solved for the bounding box dimensions of an intercepted ellipse at a given distance,  $z$ , from the projection plane.

$$A'x'^2 + C'y'^2 + E'x'z + D'y'z + Fz^2 = 1 \quad (38)$$

Note that this rotation is about an  $xy$  plane normal vector. As in Equation (20), the values of the primed coefficients are functions of the original coefficients and the rotation angle.

In this rotated frame, the intercepted ellipse defines the extremes, in the  $x'$  and  $y'$  directions, that the ellipsoid extends in the intercepting  $xy$  plane (Fig. 6). As can be seen in Figure 6, if one of the ellipse centroid coordinates is substituted into the rotated equation, one can solve the equation for the other coordinate to uniquely define the tangent point. When this is done for both centroid coordinates independently the bounding box of the intercepted ellipse is defined.

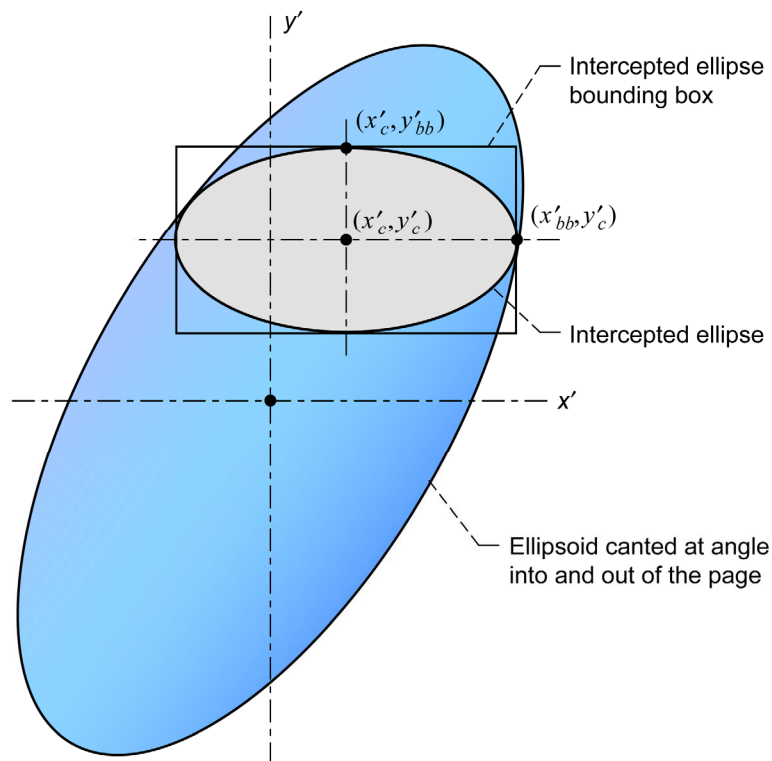


Figure 6.—Illustration of an ellipsoid sectioned by a plane after rotation of axes to eliminate the cross term. Maximum extent of the intercepted ellipse is defined by the points  $(x'_bb, y'_c)$  and  $(x'_c, y'_bb)$  where  $(x'_c, y'_c)$  is the centroid of the intercepted ellipse.



The centroid of the intercepted ellipse can be found by taking the partial derivative of the rotated equation with respect to  $x'$  and solving for  $x'$  and then with respect to  $y'$  and solving for  $y'$ , with the following result

$$x'_c = \frac{-D'z}{2A'} \quad y'_c = \frac{-E'z}{2C'} \quad (39)$$

Note the similarity to Equations (33) and (36) with  $B' = 0$ .

If  $x'_c$  is substituted into the rotated equation

$$A' \left( \frac{-D'z}{2A'} \right)^2 + C'y'^2 + E' \left( \frac{-D'z}{2A'} \right) z + D'y'z + Fz^2 = 1 \quad (40)$$

solving for  $y'$  gives the  $y'$  maximum distances

$$y'_1, y'_2 = \left[ \begin{array}{c} \frac{-E'z + \sqrt{\frac{1}{A'} \left[ z^2 (A'E'^2 + C'D'^2) - 4A'C'(Fz^2 - 1) \right]}}{2C'} \\ \frac{-E'z - \sqrt{\frac{1}{A'} \left[ z^2 (A'E'^2 + C'D'^2) - 4A'C'(Fz^2 - 1) \right]}}{2C'} \end{array} \right] \quad (41)$$

The difference between these solutions is the  $y'$  bounding box dimension,  $y'_{bb}$ .

$$y'_{bb} = \frac{\sqrt{\frac{1}{A'} \left[ z^2 (A'E'^2 + C'D'^2) - 4A'C'(Fz^2 - 1) \right]}}{C'} \quad (42)$$

Note that this Equation (42) is a function of  $z$ , the distance of the sectioning plane from the ellipsoid centroid.

Starting with the other coordinate of the intercepted ellipse centroid, the  $x'$  maximum distances are

$$x'_1, x'_2 = \left[ \begin{array}{c} \frac{-D'z + \sqrt{\frac{1}{C'} \left[ z^2 (A'E'^2 + C'D'^2) - 4A'C'(Fz^2 - 1) \right]}}{2A'} \\ \frac{-D'z - \sqrt{\frac{1}{C'} \left[ z^2 (A'E'^2 + C'D'^2) - 4A'C'(Fz^2 - 1) \right]}}{2A'} \end{array} \right] \quad (43)$$

and the  $x'$  bounding box dimension,  $x'_{bb}$ , is

$$x'_{bb} = \frac{\sqrt{\frac{1}{C'} \left[ z^2 (A'E'^2 + C'D'^2) - 4A'C'(Fz^2 - 1) \right]}}{A'} \quad (44)$$

The larger of the two solutions for the  $x'$  and  $y'$  intercepts relate the maximum extents of the intercepted ellipse in the  $x'$  and  $y'$  directions in terms of the sectioning plane distance,  $z$ . Taking the derivative of these equations with respect to  $z$ , setting to zero and solving for  $z$  will give the distance from the projection plane where the distance from a  $xy$  plane normal through the ellipsoid centroid reaches its maximum in the  $x'$  and  $y'$  directions.

First taking the derivative of the larger of the two  $y'$  bounding box solutions

$$\frac{d}{dz} \frac{-E'z + \sqrt{\frac{1}{A'} \left[ z^2 (A'E'^2 + C'D'^2) - 4A'C'(Fz^2 - 1) \right]}}{2C'} \quad (45)$$

Solving the derivative symbolically and setting to 0

$$\frac{-E' + \frac{z(A'E'^2 + C'D'^2 - 4A'C'F)}{\sqrt{A' \left[ z^2 (A'E'^2 + C'D'^2) - 4A'C'(Fz^2 - 1) \right]}}}{2C'} = 0 \quad (46)$$

Solving for  $z$  gives two roots

$$z_1, z_2 = \left[ \frac{\frac{2A'E'}{\sqrt{C'(D'^2 - 4A'F)^2 + A'E'^2(D'^2 - 4A'F)}}}{-2A'E'} \right] \quad (47)$$

Back substituting the positive root of  $z$  in Equation (47) into the larger of the two  $y'$  maximum distance Equations (41) and simplifying gives

$$y'_p = \sqrt{\frac{4A'F - D'^2}{4A'C'F - A'E'^2 - C'D'^2}} \quad (48)$$

which is the  $y'$  projected ellipse bounding box half-dimension. Using a similar methodology, starting with the larger  $x'$  maximum distance solution Equation (43), gives

$$x'_p = \sqrt{\frac{4C'F - E'^2}{4A'C'F - A'E'^2 - C'D'^2}} \quad (49)$$

which is the  $x'$  projected ellipse bounding box half-dimension.

Given the center of the projected ellipse (which coincides with the ellipsoid centroid) and the projected ellipse bounding box touching points, one can find the equation of the projected ellipse (two points on the ellipse with an additional constraint and three unknowns). The coefficients  $A_p$ ,  $B_p$ , and  $C_p$  for the projected ellipse equation ( $A_p x'^2 + B_p x'y' + C_p y'^2 = 1$ ) are

$$A_p = A' - \frac{D'^2}{4F} \quad (50)$$

$$B_p = \frac{-D'E'}{2F} \quad (51)$$

$$C_p = C' - \frac{E'^2}{4F} \quad (52)$$

As the projected ellipse is in the  $z = 0$  plane, the  $E_p$ ,  $D_p$ , and  $F_p$  terms are zero. Using these values for  $A_p$ ,  $B_p$ , and  $C_p$  and Equations (18) and (21) (with  $F = -1$ ), one can calculate the projected ellipse orientation and dimensions. Once the dimensions and orientation of the projected ellipse are known one can use Equation (28) (sectioned ellipse area) to calculate the sectioned area of a surface connected projected ellipse.

## Observations

In working with the above derivations, the following may be observed about ellipsoid sections

- (1) An arbitrary planar section of an ellipsoid is an ellipse.
- (2) Any ellipsoid will always have at least two orientations that will have circular cross sections.
- (3) For any arbitrarily oriented ellipsoid, if the sectioning plane is translated along the plane's normal vector, the orientation of the intercepted ellipse remains constant.
- (4) The maximum cross sectional area of an ellipsoid for a given orientation is always a section through its centroid.
- (5) For any fixed orientation, the projected area of an ellipsoid will always be greater than, or equal to, the maximum cross sectional area.
- (6) At most three rotations of axes are required to arrive at an arbitrary orientation for a general ellipsoid and there are twelve possible orthogonal rotation sequences (Ref. 3).

## Summary

The solutions presented were derived to facilitate modeling of the distribution of measurable inclusion dimensions in metallic materials. Inclusions are assumed to be well approximated as randomly oriented general (three independent semi-axes) ellipsoids. These solutions may also be applied for applications where ellipsoidal shaped particles are embedded in a matrix and the sectioned area of the particles is to be calculated/simulated. These applications could include: dispersed particulates, aggregates, closed cell foams, and secondary phase grains in complex alloys.

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14. ABSTRACT Nonmetallic inclusions in gas turbine disk alloys can have a significant detrimental impact on fatigue life. Because large inclusions that lead to anomalously low lives occur infrequently, probabilistic approaches can be utilized to avoid the excessively conservative assumption of lifing to a large inclusion in a high stress location. A prerequisite to modeling the impact of inclusions on the fatigue life distribution is a characterization of the inclusion occurrence rate and size distribution. To help facilitate this process, a geometric simulation of the inclusions was devised. To make the simulation problem tractable, the irregularly sized and shaped inclusions were modeled as arbitrarily oriented, three independent dimensioned, ellipsoids. Random orientation of the ellipsoid is accomplished through a series of three orthogonal rotations of axes. In this report, a set of mathematical models for the following parameters are described: the intercepted area of a randomly sectioned ellipsoid, the dimensions and orientation of the intercepted ellipse, the area of a randomly oriented sectioned ellipse, the depth and width of a randomly oriented sectioned ellipse, and the projected area of a randomly oriented ellipsoid. These parameters are necessary to determine an inclusion's potential to develop a propagating fatigue crack. Without these mathematical models, computationally expensive search algorithms would be required to compute these parameters.					
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